SCIENCE

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HENRI POINCARÉ AS A MATHEMATICAL PHYSICIST[1]

WHEN I was asked by the secretary to contribute a paper of general interest before this section I was overwhelmed with the sense of my inability to do so, but when he suggested that I should take as a subject the work of Henri Poincaré as a mathematical physicist, I consented, because, however slight might be my capability, the subject was a most congenial one. The great Frenchman whose untimely death at the age of 58 the whole scientific world deplores was a man of extraordinary versatility, while his productiveness is measured by the fact that the total number of separate contributions from his pen reaches nearly the sum of a thousand. France has always known how to honor her great men, even if she does not understand them, and the impression produced by the death of Poincaré on the whole country was profound. The news was communicated to me in London at the celebration of the Royal Society by his friend and distinguished colleague, Émile Picard, who in a voice choked with emotion pronounced the words, "Poincaré est mort!"

While there can be no doubt that the greatest work of Poincaré consisted in his work in pure analysis, we must not forget that for ten years he filled the chair of mathematical physics of the Faculté des Sciences. During this time he touched every conceivable part of the subject and it may be truly said that he touched nothing that he did not adorn. Fourteen volumes

[1] Read before Section A of the American Association for the Advancement of Science, December 31, 1912.

of published lectures attest his skill as a teacher, the names of which I will not take your time to rehearse, merely remarking that in addition to the usual treatments of electricity, optics, the conduction of heat, thermodynamics, capillarity, elasticity and hydrodynamics there are several volumes on the modern subjects of electrical oscillations and the interrelations of electricity and optics.

The work of the mathematical physicist is of two sorts, according as the emphasis is laid on the word physics or on the word mathematical. In the latter case the investigator concentrates his attention upon the attempt to demonstrate that certain problems have solutions, furnishing so-called existence theorems. In the former the attempt is made to find the solutions, assuming that they exist, in a form suitable for numerical computation. Poincaré did both, and, although capable of the highest flights into abstract mathematics, was by no means insensible to the needs of the practical man, meaning by that not only the physicist, but even the telegraph engineer. This is attested by the number of articles that he wrote on the theory of telegraphy, both with and without wires, as well as by the courses of lectures that he gave at the higher professional school of posts and telegraphs. It is certainly a very rare thing for a pure mathematician of the highest ability to write an article on the theory of the telephone receiver, yet this was done by Poincaré, while in a paper on the propagation of current in the variable period on a line furnished with a receiver he attacked an almost untouched field of very great mathematical importance in the theory of differential equations.

What is particularly striking in all of Poincaré's writings is not so much the clearness of exposition or the elegance of arrangement, for his lectures possess many of the faults of lectures published by students and his short articles are often extremely difficult reading, but rather the remarkable directness with which he proceeds to his results and the extraordinary command of every resource of pure mathematics, particularly of Cauchy's theory of the functions of a complex variable. I know of but one other author whose resources in function-theory seem to be at all comparable with those of Poincaré, I mean Professor Sommerfeld, who seems to be able to communicate his powers in that line to his students. The heart of mathematical physics is, without doubt, composed of partial differential equations, and in this subject Poincaré was, of course, a master. It is in connection with the definite integrals appearing in their solutions that there is great opportunity for the application of function-theory. The great art in mathematical physics is that of making approximations and it is here that Poincaré was particularly strong. It is frequently not so difficult to obtain the solution of the differential equation as to interpret its physical meaning. In this matter Poincaré reminds us of his great countryman Cauchy.

I shall not attempt to make an analysis of the articles of Poincaré, many of which I have great difficulty in following and many of which could be far better treated by others here present. I shall merely undertake to give a slight idea of the contents of those which have particularly impressed me. I presume his contributions of most far-reaching importance from a mathematical point of view are his articles on the equations of mathematical physics, of which he wrote three. This is a subject which has received an enormous amount of attention during the last twenty-five years and it may be undoubtedly said that in

this work Poincaré's contributions were fundamental. His first article "Sur les Equations aux Dérivées Partielles de la Physique Mathématique" appeared in the *American Journal of Mathematics* in 1890. The equations of mathematical physics are all very similar and may be practically all reduced to three or four. Of these the equation of Laplace,

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V}{\delta z^2} = \Delta V = 0,$$

is the most important. The so-called boundary problem of finding a solution of Laplace's equation, valid for a certain region of space, that shall take prescribed values at the surface bounding this space is known as Dirichlet's problem. Of this the problem of the distribution of electricity on the surface of a conductor is a particular case, the function given on the surface reducing to a constant.

The latter example is a case of the outside problem, where in addition we have the condition that the desired function must vanish at infinity. The demonstration of the existence of such a function given by Riemann and depending upon the application of the calculus of variations to the definite integral

$$\iiint \left[\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2\right] dxdydz$$

is lacking in rigor and the attempt to replace it has engaged the attention of some of the greatest mathematicians. In the present paper Poincaré gives a new method of great universality for proving the so-called Dirichlet principle. It depends upon the fact that the boundary problem can be exactly solved for the sphere and also upon the theorem discovered by Green that a potential function due to attracting masses lying within a closed surface may be exactly imitated by placing the masses in a surface distribution on the surface of the sphere. This Poincaré calls the *balayage* of the sphere, the masses being swept out of the interior and deposited on the surface. For any surface to be treated the space within is filled up by an infinite number of spheres such that any point within the given surface lies in at least one of them, and these spheres are swept in a certain order so that the process is a convergent one. The principle of Dirichlet is thus established, but a practical method of finding the solution is not given. The other equation considered in this paper is Fourier's equation for the conduction of heat,

$$\frac{\delta V}{\delta t} = a^2 \Delta V.$$

In this case the boundary condition is not as simple as in Dirichlet's problem, but we have at the surface,

$$\frac{\delta V}{\delta n} + hV = 0,$$

where h is called the emissivity of the body. In this case it is demonstrated by the aid of the calculus of variations that the problem is possible, the demonstration being that of the existence of an infinite series of functions U_n satisfying the conditions that on the surface of the body

$$\frac{\delta U_n}{\delta n} + hU_n = 0$$

and in its interior

$$\Delta U_n + k_n U = 0,$$

where the numbers $k_1, k_2 \cdots k_n$ are positive constants such that

$$k_1 < k_2 < k_3 \cdots .$$

Physically these functions have the property that if the temperature of the body at a given instant is distributed according to any one of them then this distribution will remain unchanged during all subsequent time, merely dying away at an exponential rate. It is interesting to notice that in the last part of this paper Poincaré compares his process to that used by

Fourier in deducing his equation; namely, by supposing the body to be composed of a large number of small bodies each radiating heat to all the others according to the law that the amount of heat radiated in a given time is proportional to the difference of temperature of the two bodies. Thus a system of ordinary differential equations is arrived at,

$$\frac{dV_i}{dt} + \sum_{k=1}^{k=n} C_{ik}(V_i - V_k) + C_i V_i = 0, \quad i=1, 2 \cdots n,$$

which is readily solved by putting

$$V_i = U_i e^{-\lambda t},$$

in which case the differential equations become algebraic linear equations for the quantities U_i,

$$\lambda U_i = \sum_k C_{ik}(U_i - U_k) + C_i U_i.$$

In order to solve them it is necessary that the determinant

$$\Delta = \begin{vmatrix} C_1 - \lambda, & -C_{12}, & -C_{13}, & \cdots \\ -C_{12}, & C_2 - \lambda, & -C_{23}, & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

should vanish. But we get the same equations if we consider the quadratic form

$$\Phi \equiv \Sigma C_{ik}(V_i - V_k)^2 + \Sigma C_i V_i^2,$$

which being equated to a constant represents an ellipsoid in n-dimensional space. The equations for the axes are our linear equations. The axes of this ellipsoid being all real, all the roots of the determinant Δ are real. The form may then be decomposed into a sum of squares.

$$\Phi = \lambda_1 \phi_1^2 + \lambda_2 \phi_2^2 \cdots, \text{ where}$$
$$\phi_p = U_{p1} U_1 + U_{p2} U_2 \cdots.$$

Upon the properties of this quadratic form depends the whole theory. When the number of particles becomes infinite the system of ordinary differential equations leads in the limit to Fourier's partial differential equation, and the theorems which will arise if the passage to the limit is justified lead to Poincaré's deduction. It is to be noticed that this principle had been used before by Lord Rayleigh in connection with the theory of vibrations and the possibility of passing to the limit postulated by him is now known as Rayleigh's principle. More interesting still is the fact that to-day this process used by Rayleigh and Poincaré has become in the hands of Fredholm and Hilbert a rigorous method, that of integral equations, which is at present occupying a large part of the attention of the mathematical world.

In his second paper on the same subject published in 1894 in the *Rendiconti del Circolo Mathematico di Palermo,* Poincaré passes to the consideration of the more general equation

$$\Delta u + \xi u + f = 0,$$

where ξ is a constant and f a given space-function. This equation includes not only Fourier's equation but the equation of waves

$$\frac{\delta^2 \phi}{\delta t^2} = a^2 \Delta \phi + g$$

if we put

$$\phi = e^{int} u, \quad \xi = \frac{n^2}{a^2}.$$

Regions in which f is not zero are called sources of heat or sound. Poincaré proceeds in this equation to develop u according to powers of ξ, as had been done by Schwarz, thus obtaining a solution by successive approximations which he proves to be convergent. He also proves the fundamental property that a solution of the equation is a meromorphic function of the parameter ξ having an infinite number of simple poles, that is to say,

$$U = \Sigma \frac{\alpha_i U_i}{\xi - k_i} \quad \text{where} \quad \Delta U_i + k_i U_i = 0.$$

This theorem is also fundamental in the theory of integral equations. To speak in the language of sound Poincaré demonstrates the existence of an infinite number of natural vibrations for the air in a cavity surrounded by the given surface, the characteristic numbers k_i or values of the poles

of the solution giving us their periods, and the nature of the function U showing the phenomenon of resonance, that is to say, the vibration becoming infinite when the impressed force has the period of one of the natural vibrations. The method of Poincaré leads directly to Schmidt's solution of the integral equation.

In a third paper published in the *Acta Mathematica* in 1897 Poincaré deals with what he calls Neumann's problem, which he defines as follows: To find a potential of a double layer whose limiting values inside and outside the surface are denoted by V and V', and which satisfies the equation at the surface,

$$V - V' = \lambda(V + V') + 2\Phi,$$

where λ is a parameter. If $\lambda = -1$ this reduces to the interior Dirichlet's problem $V = \Phi$ and if $\lambda = 1$ to the exterior problem $V' = -\Phi$. By means of a development in powers of the parameter λ a solution is obtained by successive approximations which is proved to converge. One of the most important results of this paper is the demonstration of the existence of a series of what he calls fundamental functions which have the property of being potentials of simple layers, and

$$\frac{\delta \Phi_i}{\delta u} = -\lambda_i \frac{\delta \Phi'}{\delta n}$$

in terms of which he deems it probable that any function on the surface may be developed, so that when these functions are known Dirichlet's problem may be solved. These reduce for the sphere to spherical harmonics and for the ellipsoid to Lamé's functions, and they are the characteristic functions belonging to integral equations.

Let us now turn to a different field. In 1893 attention was called by Poincaré to an equation which has become famous, called by him the equation of telegraphists. This equation

$$a\frac{\delta^2 u}{\delta t^2} + 2b\frac{\delta u}{\delta t} + cu = \frac{\delta^2 u}{\delta n^2}$$

had been introduced before by Kirchhoff and Heaviside, but its physical interpretation had not been emphasized. If the first term is lacking it reduces to Fourier's equation and it had been shown by Sir William Thomson in 1855 that signals were propagated through a submarine cable in accordance with it. If the second and third terms are absent the equation reduces to the equation of sound in one dimension and shows the propagation of waves unchanged in form with a constant velocity. The equation of telegraphists may then be expected to combine the properties of transmission in waves and heat transmission with an infinite velocity. The first term arises from the consideration of the self-induction of the line neglected by Thomson and the second term from the resistance which can generally not be neglected. By the simple method of the assumption that u can be represented as a Fourier's integral, after taking out an exponential factor, so that

$$\frac{\delta^2 U}{\delta t^2} = \frac{\delta^2 U}{\delta x^2} + U, \quad U = \int_{-\infty}^{\infty} \theta(q) e^{iqx} dq,$$

Poincaré obtains the solution

$$U = \int_{-\infty}^{\infty} e^{iqx}\left[\theta \cos t\sqrt{q^2-1} + \theta_1 \frac{\sin t\sqrt{q^2-1}}{\sqrt{q^2-1}}\right] dq,$$

which he shows by an application of the theory of functions to depend upon a Bessel function. The remarkable physical result is that while the disturbance, like the sound wave, is propagated with a finite velocity, after it has passed over a given point it leaves a residue or trail which gradually dies away like heat. In a later paper he discussed the effect on the telegraphist's equation of terminal conditions of a complicated sort necessitated by the employment of receiving apparatus.

Probably the favorite subject in mathe-

matical physics treated by Poincaré was that of electrical waves and oscillations. The reason for this is not far to seek. Not only are the equations of Maxwell's theory, complicated though they seem to be and involving a large number of vectors, capable of reduction to one of the forms already mentioned, but their solution calls for great knowledge of differential equations and even in the last few years in the hands of Poincaré permits of being treated by integral equations. Besides this the applications to the subject of wireless telegraphy are of great practical importance as well as theoretical interest. If we write Maxwell's equations in the simplest case,

$$\alpha = \frac{\delta H}{\delta y} - \frac{\delta G}{\delta z},$$

$$\cdots\cdots\cdots \qquad \frac{\delta f}{\delta x} + \frac{\delta s}{\delta y} + \frac{\delta k}{\delta z} = \rho,$$

$$4\pi f = -\frac{\delta F}{\delta t} - \frac{\delta \psi}{\delta n},$$

$$\cdots\cdots\cdots \qquad \frac{\delta \alpha}{\delta x} + \frac{\delta \beta}{\delta y} + \frac{\delta \gamma}{\delta z} = 0,$$

$$4\pi\left(u + \frac{\delta f}{\delta t}\right) = \frac{\delta \gamma}{\delta y} - \frac{\delta \beta}{\delta z},$$

$$\cdots\cdots\cdots$$

where (α, β, γ) denotes the magnetic field, 4π $(f, g, h,)$ the electric field, u, v, w the conduction current, ρ the electric density, Ψ the electric scalar potential, (F, G, H) the vector-potential, where the vector-potential is defined by the differential equations

$$\alpha = \frac{\delta H}{\delta y} - \frac{\delta G}{\delta z}, \quad \cdots$$

there is still something indeterminate in this vector. Maxwell assumes

$$\frac{\delta F}{\delta x} + \frac{\delta G}{\delta y} + \frac{\delta H}{\delta z} = 0,$$

which does not lend itself to simplicity, but if instead we put, as was later done by Lorentz,

$$\frac{\delta F}{\delta x} + \frac{\delta G}{\delta y} + \frac{\delta H}{\delta z} + \frac{\delta \psi}{\delta t} = 0,$$

we get a great simplification. It is notable that this was done by Poincaré in 1893 in his lectures on electrical oscillations, evidently quite independently of Lorentz. Our equation for the vector-potential then reduces to the form

$$\frac{\delta^2 F}{\delta t^2} - \Delta F = 4\pi u,$$

which shows that if there is no current anywhere the vector-potential is propagated in waves. The scalar potential also satisfies the equation

$$\frac{\delta^2 \psi}{\delta t^2} - \Delta \psi = 4\pi\rho.$$

It was shown by Lorentz in 1892 and independently in the same year by Beltrami, that this equation, which, in case the first term vanishes, reduces to that of Poisson, is satisfied by a potential function

$$\psi = \iiint \frac{\rho' d\tau}{r},$$

in which, however, ρ' represents the value of ρ not at the instant in question but at an instant precedent by the time required to come from the point of integration with the wave-velocity. Such a potential is now known as a retarded potential and I may be allowed for a moment to digress upon the interesting history of this Lorentz-Beltrami equation. Its properties with those of the retarded potential are given by Poincaré in his lectures in 1893, evidently quite independently of Lorentz. It turns out, however, that the properties of the equation are given in Lord Rayleigh's "Theory of Sound," appearing in 1877. This, however, is not its first appearance, for we find the same equation in a paper by L. Lorenz, of Copenhagen, in 1869, on elasticity. More remarkable yet in a paper presented by Riemann to the Royal Society of Göttingen in 1858, but published only in 1867 after Riemann's death, we find the same identical equation for the electric potential although deduced from considerations which are now untenable. It is curious to remark that although this equation is mentioned by Maxwell in the last chapter

of his great work he does not commit himself as to its conclusions, but states that Clausius has shown that the hypothesis that the potential is propagated like light does not lead to the known laws of electrodynamics. Curiously enough, to-day this is exactly what we do believe, and it is interesting to know that such a result was vainly sought for by Gauss.

It is easy to conceive, the equations of electrical propagation being so similar to those of the propagation of sound waves, how the question of fundamental functions arises in connection with electrical oscillations emitted by a conductor of given form. The only case of anything except a linear conductor that has been completely treated is that of a sphere and of this a treatment was given in the same lectures in 1893 by Poincaré. One of the most important questions in wireless telegraphy has been during the last ten years and still is the explanation of the possibility of sending Hertzian waves across the Atlantic, a distance of perhaps one tenth of the way around the earth. The question of diffraction has always been an attractive one and in the case of electric waves makes great demands upon the powers of the mathematician. After a number of articles on the subject Poincaré in 1909 applies to it the method of integral equations, which he continues in a lecture on the Wolfskehl foundation at Göttingen, and later in a tremendous paper in the *Palermo Rendiconti*.

The development of Maxwell's electromagnetic theory that has taken place in the last twenty-five years has led to a theory that has attracted the greatest interest among mathematical physicists and has, in fact, become in certain parts of the world no less than a mania. I refer to the so-called principle of relativity, a name which was given to it first, if I am not mistaken, by Poincaré. This principle is no less than a fundamental relation between time and space, intended to explain the impossibility of determining experimentally whether a system, say the earth, is in motion or not. In an elaborate paper published in 1905 in the *Palermo Rendiconti* entitled, "Sur la dynamique de l'électron," he defines the principle of relativity by means of what he calls the Lorentz transformation. If the coordinates and the time receive the following linear transformation,

$$x' = kl(x + \epsilon t), \quad t' = kl(t + \epsilon x), \quad y' = ly,$$
$$z' = lz, \quad k = \frac{1}{\sqrt{1 - \epsilon^2}},$$

the function $x^2 + y^2 + z^2 - t_1^2$ and the equations of electric propagation will remain invariant. From this follows the impossibility of determining absolute motion. Poincaré then submits the Lorentz transformation, which he shows belongs to a group, to an examination with regard to the principle of least action, which he shows holds for the principle of relativity. He further shows that by the aid of certain hypotheses gravitation can be accounted for and shown to be propagated with the velocity of light. This is a subject which is now very much in the air, but it must be said that various writers arrive at conflicting results.

From what I have said, it will have been seen that Poincaré was exceedingly up-to-date and at once made the newest speculations and theories his own. As the final example of this may be named a theory which has created nearly as great a shock as that of relativity. I mean the theory of light *quanta* introduced by Planck to account for the laws of radiation from a hot body. In order to apply the laws of probability to electric resonators Planck had felt obliged to introduce the hypothesis that energy is emitted by resonators not in continuous amounts but in amounts depending upon certain multiples of a definite quantity

known as the quantum. In one of his very last papers published in January, 1912, in the *Journal de Physique,* Poincaré submits the theory of quanta to a searching examination and as a conclusion announces that it is impossible to arrive at Planck's law except under the assumption that resonators can acquire or lose energy only in discontinuous amounts. If this is true we have an extraordinary departure from received ideas and it will be necessary to suppose that natural phenomena do not obey differential equations.

Enough has been said to show the extraordinary variety of the subjects treated by this commanding intellect in the subject of mathematical physics alone. In repeating what I stated at the outset that the striking quality displayed by Poincaré is his extraordinary skill in analysis, I do not mean for a moment to imply anything against his intense receptivity for all physical ideas, for which he had a very great penetration. It is true that he sometimes met severe criticism from physicists. In particular Professor Tait made a bitter attack on his treatise on thermodynamics, but in my opinion Poincaré was well able to defend himself. It has sometimes been doubted whether he thoroughly appreciated Maxwell's ideas as to the theory of electricity, but this is of small moment, seeing that he so well understood their consequences. It must be said that Poincaré was not one who contributed fundamental new ideas to our stock of physical conceptions, such as the ideas put forth by Carnot, Kelvin, Maxwell, Lorentz with his principle of local time or Planck with his quanta.

I may in conclusion be permitted to state my opinion that the best persons to appoint to chairs of mathematical physics and those most likely to enrich our conceptions are those who have themselves had experience in dealing with nature with their own hands in the laboratory, and who may be expected to have more feeling for her modes of action than skill in analysis. Thus I believe Helmholtz, Kelvin, Maxwell, and Lord Rayleigh to have been more important contributors to mathematical physics than Poincaré, but this is not to say that the latter was not an intellect of superlative greatness.

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UNIVERSITY ORGANIZATION[1]

This subject has become in recent years one of intense interest. In most utterances on the subject the prominent feature is the statement that our universities are undemocratic, that they are monarchical institutions in a democratic country. This criticism takes various forms. When a university president speaks, the shortcomings of the university are due to the fact that the governing board are ignorant, shallow-minded, arrogant and headstrong; that they insist upon deciding matters beyond their knowledge and will not be guided by the president. When a university professor speaks it is the university presidency which is at fault. Autocracy, blindness, willfulness, prejudice, partiality, lofty-mindedness, oratorical ability, money-getting talents, piety and many other virtues and vices are ascribed to our presidents, but in the minds of nearly all writers the presidency is an unsatisfactory tool. When an outsider speaks, both president and governing board are parts of a vicious organization.

Let us grant that there is much truth in this. Boards may be unwise; the presidency may be unequal to its responsibili-

[1] With especial reference to state universities. An address delivered before a body of university men at Minneapolis, November 10, 1913.